# On shape stability for a storage model

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**Abstract**

We consider stability of shape for a storage model on $n$ nodes. These nodes support $\mathcal{K}$ neighborhoods $S_i \subset \{1, \ldots, n\}$ and items arrive at the $S_i$ as independent Poisson streams with rates $\lambda_i$, $i = 1, \ldots, \mathcal{K}$. Upon arrival at $S_i$ an item is stored at node $j \in S_i$ where $j$ is determined by some policy. Let $X_j(t)$ denote the number of items stored at $j$ at time $t$ and let $X(t) = \big(X_1(t), \ldots, X_n(t)\big)$. Under natural conditions on the $\lambda_i$ we exhibit simple local policies such that $X(t)$ is positive recurrent (stable) in shape.

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## 1 The model

We consider a storage system (or library) with a finite number of nodes where identical items are to be stored. The $n$ nodes support non-empty neighborhoods $S_i$, $i = 1, \ldots, \mathcal{K}$ with

$$\bigcup_{i=1}^{\mathcal{K}} S_i = \{1, \ldots, n\},$$

and $1 \leq \mathcal{K} \leq 2^n - 1$. Items arrive at the neighborhoods as independent Poisson processes with rates $\lambda_i > 0$ at $S_i$, $i = 1, \ldots, \mathcal{K}$ where we suppose that $\sum_{i=1}^n \lambda_i = 1$. Upon arrival at $S_i$ an item is stored at a node $j \in S_i$ where $j$ is chosen by some policy. We consider *local* Markov policies where each allocation decision is a function of the state, at the arrival time of the item, of the neighborhood where the item arrives. We will make this more precise below.

Stability in shape is of interest in several models. There are of course various growth models, see for example the crystal growth model studied in [1], though the methods used there are very different from those we use in this paper. Another model which is relevant is a queueing system with server vacations or maintenance periods where stability in shape can be seen as a fairness criterion for arriving jobs. It is also reasonable to view this storage model as a simplified version of the supermarket model (by dropping the service), see for example [6].

We have chosen to focus on the routing aspect of this model here. Rather more complex phenomena appear when service is considered as well and we are investigating a model in which service times are dependent upon both the arrival neighborhood and the allocated server.

We return to the description of our model. Let $|S_i| = \kappa_i$ denote the size of neighborhood $i$ and suppose the nodes in $S_i$ are enumerated in some way, so that $S_i = \{s_1^i, \ldots, s_{\kappa_i}^i\}$.

**Definition 1.1.** *We say that $j, k \in \{1, \ldots, n\}$ are neighbors (and write $j \sim k$), if $j$, $k \in S_i$ for some $i$.*

This equivalence relation can be used to define the graph $\mathcal{G}$ with vertices $\{1, \ldots, n\}$ and edges $\mathcal{E}$, where $w = \langle j, k \rangle \in \mathcal{E}$ iff $j \sim k$. Our main result (Theorem 2.1) needs the following assumption.

**Condition 1.1.** *The graph $\mathcal{G}$ is connected.*

If the Condition 1.1 is not fulfilled, then we have two or more disconnected components, that is, sets of nodes such that arrivals to one of these sets cannot be routed to the other. In this case, it is impossible to obtain positive recurrence in shape, for any routing policy. If the number of disconnected components is at least 4, then even null-recurrence is impossible. The reasoning for this is similar to that in the argument immediately after Theorem 2.1.

Denote the configuration of the system at moment $t$ by

$$X(t) = (X_1(t), \ldots, X_n(t)),$$

where $X_i(t)$ is the number of items stored at node $i$ at time $t$. The center of mass or average load of the configuration is

$$M(t) = \frac{1}{n} \sum_{i=1}^{n} X_i(t),$$

and we denote the *shape* of the configuration by

$$\tilde{X}(t) = \big(\tilde{X}_1(t), \ldots, \tilde{X}_n(t)\big) = \big(X_1(t) - M(t), \ldots, X_n(t) - M(t)\big),$$

the vector of loads relative to the center of mass. Note that, if a new item arrives at time $t$, then $M(t) = M(t-) + \frac{1}{n}$. Also, if we know the shape $\tilde{X}(t)$, it implies that we know which node is minimally loaded and we know the load differences between the nodes (as $X_i(t) - X_j(t) = \tilde{X}_i(t) - \tilde{X}_j(t)$).

Obviously, the process $X(t)$ is Markovian for any decision rule that depends only on the current node loads. In order for the process $\tilde{X}(t)$ to be

Markovian, we require that the decision of choosing the node is made accordingly to some decision rule which depends only on the current *shape* of the system. Also, we are mainly interested in *local* decision rules, that is, if an item arrives to the set $S_i$, then *the only information about the configuration of the system that can be used to make a decision is what happens in the set* $S_i$. For example, the decision can be based on the differences $\tilde{X}_l(t) - \tilde{X}_j(t)$, $l, j \in S_i$.

If the decision rule is configuration independent and time homogeneous this gives rise to a space homogeneous $(n-1)$-dimensional random walk, which is transient for $n > 3$ and at best null recurrent for $n \leq 3$. Therefore, if one wants positive recurrence in shape, the decision rule must depend on current configuration. Of course, all nodes must receive arrivals for ergodicity in shape to be achieved, hence the walk cannot live in a lower dimensional sub-space. So, our goal is to find a rule for redistributing the arriving items at each moment of time in a way to have positive recurrence in shape. One of the possible choices is to send the item to the node with minimal load $S_i$ (Join the Shortest Queue routing policy).

We present four routing policies. Two ensure the same rate of the arrivals to different nodes, and the two others guarantee stability in shape, if some explicit conditions are fulfilled. We note also that the conditions we refer to can be easily checked in practice and the implementation of routing policies we propose is algorithmically simple.

The paper is organized as follows. In Section 2 we introduce the notations, define the routing policies and state the results. In Section 3.1 we formulate the known facts we will use in our proofs. In Section 3.2, we prove Theorem 2.1, for which we need some auxiliary lemmas, and then we prove Theorem 2.2. In Section 3.3, we first prove a lemma that translates the condition of Theorem 2.3 into the language of convex analysis, and prove Theorem 2.3.

## 2 Results

Let us first introduce some notation. For $i = 1, \ldots, \mathcal{K}$ denote by $\Lambda_i$ the set of points $p^{(i)} = (p_1^{(i)}, \ldots, p_{\kappa_i}^{(i)}) \in \mathbb{R}^{\kappa_i}$ such that

$$\begin{cases} p_j^{(i)} \geq 0 & \text{for } j = 1, \ldots, \kappa_i, \\ \sum_{j=1}^{\kappa_i} p_j^{(i)} = 1. \end{cases} \tag{2.1}$$

We say that $p = (p^{(1)}, \ldots, p^{(\mathcal{K})}) \in \mathbb{R}^{\kappa_1 + \cdots + \kappa_{\mathcal{K}}}$ is *non-negative* if

$$p_j^{(i)} \geq 0 \qquad \text{for } j = 1, \ldots, \kappa_i \text{ and } i = 1, \ldots, \mathcal{K},$$

and *positive* if

$$p_j^{(i)} > 0 \qquad \text{for } j = 1, \ldots, \kappa_i \text{ and } i = 1, \ldots, \mathcal{K},$$

By $F$ denote the linear transformation that takes a point $x \in \mathbb{R}^n$ to the point $y \in \mathbb{R}^n$ such that $y_i = x_i - \frac{1}{n}\sum_{j=1}^n x_j$ for $i = 1, \ldots, n$. In words, the point $y$ represents deviations from the center of mass for the configuration $x$. Let $\mathfrak{M} = \{y \in \mathbb{R}^n : \sum_{i=1}^n y_i = 0\}$. Note that $F(\mathbb{R}^n) = \mathfrak{M}$.

Let $\tilde{X}(t) = F(X(t))$. Since $F$ is a linear transformation and $\mathfrak{M}$ is a $(n-1)$-dimensional subspace of $\mathbb{R}^n$, we can say informally that the dimension of the process $\tilde{X}(t)$ is 1 less than the dimension of $X(t)$. Since the state space of the process $X(t)$ is $\mathbb{N}^n \subset \mathbb{R}^n$, we see that the state space of the process $\tilde{X}(t)$ is $F(\mathbb{N}^n) \subset \mathfrak{M}$.

A point $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$ represents the load of the system. By $x^{(S_i)}$ denote the load of the nodes in $S_i$. Let $\mathbf{1}$ be the vector with all ones: $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^n$.

Now we define the notion of routing policy (RP).

**Definition 2.1.** *A* routing policy $P$ *is a function that takes a configuration* $x \in \mathbb{N}^n$ *to a point* $P(x) = (p^{(1)}, \ldots, p^{(\mathcal{K})}) \in \mathbb{R}^{\kappa_1 + \cdots + \kappa_{\mathcal{K}}}$ *such that* $p^{(i)} \in \Lambda_i, i = 1, \ldots, \mathcal{K}$. *For the process* $X(t)$ *(or* $\tilde{X}(t)$*) with routing policy* $P$*, an item arriving at neighbourhood* $S_i$*, when the configuration of the system*

*is $x$, is routed to node $s^i_j$ with probability $p^{(i)}_j(x)$. The decisions are made independently for each arrival.*

For the process $\tilde{X}(t)$ to be Markovian, we suppose that all routing policies satisfy the following.

**Condition 2.1.** *The routing policy $P$ depends only on the current configuration shape, that is, for any $x$ and $y$ such that $F(x) = F(y)$, we have $P(x) = P(y)$.*

There is an equivalent way to write down this condition formally: for any $c \in \mathbb{Z}$ and any $x$ and $y$ such that $x = c\mathbf{1} + y$, we have $P(x) = P(y)$.

The decision about routing can be made using the complete information about configuration shape, or only partial information:

**Definition 2.2.** *For $i = 1, \ldots, \mathcal{K}$, denote by $P^{(i)}$ the composition of routing policy $P$ and projection on $\mathbb{R}^{\kappa_i}$, i.e. for any $x$, we have $P^{(i)}(x) = p^{(i)}$ whenever $P(x) = (p^{(1)}, \ldots, p^{(\mathcal{K})})$. We say that a routing policy $P$ is* local *if, for $i = 1, \ldots, \mathcal{K}$, the function $P^{(i)}$ depends only on the load of the nodes in $S_i$: for any $x$ and $y$ such that $x^{(S_i)} = y^{(S_i)}$, we have $P^{(i)}(x) = P^{(i)}(y)$.*

Recall that we demand that a routing policy depends only on current configuration shape. In the case of local routing policy this means that, for $i = 1, \ldots, \mathcal{K}$, the function $P^{(i)}$ depends only on current configuration shape restricted to $S_i$: for any $c \in \mathbb{Z}$ and any $x$ and $y$ such that $x^{(S_i)} = c\mathbf{1}^{(S_i)} + y^{(S_i)}$, we have $P^{(i)}(x) = P^{(i)}(y)$.

In this paper we will consider four local routing policies.

**Definition 2.3.** *An* equilibrium routing policy (ERP) *is a routing policy $P$ such that $P$ does not depend on $x$ and the resulting arrivals at all nodes are independent Poisson processes with the same rate*

$$\frac{\sum_{i=1}^{\mathcal{K}} \lambda_i}{n} = \frac{1}{n},$$

*(recall that $\sum_{i=1}^{\mathcal{K}} \lambda_i = 1$).*

**Definition 2.4.** *A* strong equilibrium routing policy (SERP) *is a routing policy $P$ such that $P$ does not depend on $x$, the resulting arrivals at all the nodes are independent Poisson processes with the same rate*

$$\frac{\sum_{i=1}^{\mathcal{K}} \lambda_i}{n} = \frac{1}{n}$$

*and $P$ is positive for all $x$.*

Let us consider the following system of linear equations:

$$\begin{cases} \sum_{j=1}^{\kappa_i} \alpha_{ij} = \lambda_i & \text{for } i = 1, \dots, \mathcal{K}, \\ \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \alpha_{ij} \delta_{\ell, s_j^i} = \frac{1}{n} & \text{for } \ell = 1, \dots, n, \end{cases} \tag{2.2}$$

where

$$\delta_{\ell,m} = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m. \end{cases}$$

**Remark 2.1.** *The system* (2.2) *is a special case of the maximum bipartite matching problem and necessary and sufficient conditions for existence of positive/non-negative solutions are well-known.*

*For each non-empty collection of neighbourhoods $J \subset \{1, 2, \dots, \mathcal{K}\}$ let $S_J = \cup_{j \in J} S_j$ and let $n_J$ denote the number of nodes in $S_J$. Then,*

$$\sum_{j \in J} \lambda_j \leq n_J / n \qquad \textit{for all } J \subset \{1, \dots, n\} \tag{2.3}$$

*is necessary and sufficient for existence of non-negative solutions to* (2.2)*. Strict inequality in* (2.3) *for all $J$ except $\emptyset$ and $\{1, 2, \dots \mathcal{K}\}$ is necessary and sufficient for the existence of positive solutions to* (2.2)*.*

*Indeed, if* (2.3) *is not satisfied, then at least one node in some $S_J$ must receive items at rate greater than $1/n$, under any routing policy. The sufficiency can be shown using maximum-flow minimum-cut method (cf., for example, [4, 8]).*

**Remark 2.2.** *Note that for any parameters of the model $S_1, \dots, S_{\mathcal{K}}$ and $\lambda_1, \dots, \lambda_n$ we have:*

- *there exists an ERP iff* (2.2) *has a non-negative solution;*

- *there exists a SERP iff* (2.2) *has a positive solution.*

*Indeed, if* (2.2) *has a non-negative/positive solution we can define* $p_j^{(i)} = \alpha_{ij}/\lambda_i$*. If we have an ERP/SERP, then* $\alpha_{ij} = \lambda_i p_j^{(i)}$ *is a non-negative/positive solution of* (2.2)*.*

We also rewrite this statement in the language of convex analysis (see Lemma 3.5).

As solving (2.2) is a problem of linear programming, the existence of SERP can be easily checked in practice.

**Example 2.1.**

- *Consider a system with* $n = 3$ *nodes and all possible neighborhoods of size* 2*,* $\lambda_1 + \lambda_2 + \lambda_3 = 1$*. Then, there exists a positive solution of the system* (2.2) *iff* $\lambda_i < 2/3$ *for* $i = 1, 2, 3$*.*

- *Similarly, for* $n = 4$ *and all possible neighbourhoods of size* 2*, there exists a positive solution of the system* (2.2) *iff* $\lambda_j < 1/2$*,* $j = 1, \dots, 6$*, and* $\sum_{j\in J} \lambda_j < 3/4$ *for all* $J$ *such that* $n_J = 3$*.*

Now we define the other two routing policies which we study. For $x \in \mathbb{N}^n$, let

$$s^i_{j_{\max}}(x) = \max\left\{ s^i_j \in S_i : \ x_{s^i_j} = \max_{l=1,\dots,\kappa_i} \{x_{s^i_l}\} \right\} \tag{2.4}$$

and

$$s^i_{j_{\min}}(x) = \min\left\{ s^i_j \in S_i : \ x_{s^i_j} = \min_{l=1,\dots,\kappa_i} \{x_{s^i_l}\} \right\}. \tag{2.5}$$

In words, for any load of the system $x \in \mathbb{N}^n$, $s^i_{j_{\min}}(x)$ is the first node in $S_i$ such that in this node the load is minimal, $s^i_{j_{\max}}(x)$ is the last node in $S_i$ such that in this node the load is maximal.

**Definition 2.5.** Join the Shortest Queue (JSQ) routing policy *is the routing policy* $P(x) = (p^{(1)}, \dots, p^{(\mathcal{K})})$*, where*

$$p_j^{(i)}(x) = \begin{cases} 1 & \text{if } s^i_j = s^i_{j_{\min}}(x), \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 2.6.** *Suppose that there exists a positive solution $\alpha_{ij}$ of* (2.2). *Let* $0 < \varepsilon < \min \alpha_{ij}$. *We define* $\varepsilon$-perturbed strong equilibrium routing policy ($\varepsilon$-PSERP) *as* $P(x) = (p^{(1)}, \ldots, p^{(\mathcal{K})})$, *where*

$$p_j^{(i)}(x) = \begin{cases} \frac{\alpha_{ij}+\varepsilon}{\lambda_i} & \text{if } s_j^i = s_{j_{\min}}^i(x), \\ \frac{\alpha_{ij}-\varepsilon}{\lambda_i} & \text{if } s_j^i = s_{j_{\max}}^i(x), \\ \frac{\alpha_{ij}}{\lambda_i} & \text{otherwise.} \end{cases}$$

*If* $\kappa_i = 1$ *(i.e., the neighborhood* $S_i$ *has size* 1*), then we have no freedom to choose probabilities and* $p_j^{(i)}(x) = 1$ *for any* $x$.

Note that in each of the four cases the routing policy can be chosen to be local. Indeed, in the case of JSQ it is clear immediately from the definition. In each of the other three cases, we first need to note that we can choose the same solution of (2.2) for all $x \in \mathbb{N}^n$, then it is easy to see that the corresponding policy is local. Moreover, in the cases of ERP and SERP it does not depend on $x$.

We study the behavior of the process $\tilde{X}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_n(t))$, where $\tilde{X}_i(t) = X_i(t) - M(t)$, $M(t) = \frac{1}{n}\sum_{i=1}^n X_i(t)$, which represents the shape of the system. In order to simplify the notation, we prefer to keep the same symbol for the process with any RP; instead when dealing with $X(t)$ or $\tilde{X}(t)$ we will state explicitly which RP is used.

Let $\{X^e(m)\}_{m\in\mathbb{N}}$ (resp. $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$) be the embedded Markov chain for the process $\{X(t)\}_{t\geq 0}$ (resp. $\{\tilde{X}(t)\}_{t\geq 0}$), obtained when we look at the system only at the moments of arrivals. Note that $\{X^e(m)\}_{m\in\mathbb{N}}$ and $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ are indeed Markov chains, as the arrivals are Poisson. Note also that $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ has period $n$ under any of the policies considered (indeed, if $\tilde{X}^e(m) = x$, we need the same number of items to arrive at every node to obtain $\tilde{X}^e(m') = x$, so we must have $m' = nl$ for some $l$). For EPR, SERP and $\varepsilon$-PSERP the process $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ is irreducible, as all nodes have positive arrival rates and thus any shape can be obtained from any other shape. The situation is more delicate for JSQ routing policy.

For example, with JSQ, if node $j$ does not belong to a neighborhood of size 1, then starting from configuration $\tilde{X}^e(0) = 0$ it is impossible to obtain configuration with $\tilde{X}^e_i(m) = x$ for all $i \neq j$ and $\tilde{X}^e_j(m) = x+2$. It important to note, however, that the configuration $\tilde{X}^e(m) = 0$ *is reachable from any configuration.*

By $\tau$ denote the time of the first return to the origin:

$$\tau = \inf\{m > 0 : \tilde{X}^e(m) = 0\}. \tag{2.6}$$

We say that

(a) $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ is transient if $\mathbf{P}(\tau = \infty \mid \tilde{X}^e(0) = 0) > 0$,

(b) $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ is recurrent if $\mathbf{P}(\tau < \infty \mid \tilde{X}^e(0) = x) = 1$ for any $x$,

(c) $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ is positive recurrent if $\mathbf{E}(\tau \mid \tilde{X}^e(0) = x) < \infty$ for any $x$.

This is of course a slight abuse of notation, since, strictly speaking, only the process restricted to the irreducible class of 0 can be (positive) recurrent. Nevertheless, we prefer to give the definition in this form because, as we will see below, (b) and (c) either hold for all or for no $x$ (so that the results are more complete).

Since the rates of our processes are bounded away from 0 and $\infty$, positive recurrence of $\{\tilde{X}(t)\}_{t\geq 0}$ is equivalent to positive recurrence of $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$. So, we will prove the results for $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$.

**Theorem 2.1.** *Suppose that Condition 1.1 is satisfied and there exists a positive solution of* (2.2)*. Suppose that we construct the process* $\{\tilde{X}^e(m)\}_{m\in\mathbb{N}}$ *using either JSQ routing policy or* $\varepsilon$*-PSERP. Then* $\tilde{X}^e(m)$ *is positive recurrent. Moreover, there exists* $c > 0$ *such that for all* $0 < c' < c$ *we have* $\mathbf{E}(e^{c'\tau} \mid \tilde{X}^e(0) = x) < \infty$ *for all* $x$*.*

Note that using ERP or SERP it is impossible to have positive recurrence of $\tilde{X}^e(m)$. Indeed, these routing policies provide independent Poisson arrivals with the same rate to all nodes. Then the behavior of the shape can

be described by a $(n-1)$-dimensional random walk with zero drift, which is transient if $n > 3$ and null-recurrent if $n \leq 3$.

Define the shape magnitude as

$$\mathfrak{D}(\tilde{X}(t)) = \sum_{i=1}^{n} \tilde{X}_i(t)^2 = \sum_{i=1}^{n} (X_i(t) - M(t))^2 \tag{2.7}$$

(so $\mathfrak{D}(\tilde{X}(t))$ is in fact the square of the Euclidean norm of $\tilde{X}(t)$).

**Remark 2.3.** *From the proof of Theorem 2.1 (equations (3.4) and (3.8)) it can be extracted that JSQ routing policy minimizes the expected shape magnitude, that is, for any routing policy we have*

$$\begin{aligned} &\mathbf{E}^{any\ RP}(\mathfrak{D}(\tilde{X}^e(m+1)) \mid \tilde{X}^e(m) = x) \\ \geq\ \ &\mathbf{E}^{JSQ}(\mathfrak{D}(\tilde{X}^e(m+1)) \mid \tilde{X}^e(m) = x). \end{aligned}$$

We also have the following converse results (in some sense) to Theorem 2.1. Note that in Theorems 2.2 and 2.3 we do not require the routing policy $P$ to be local.

**Theorem 2.2.** *Fix the parameters of the model: $S_1, \ldots, S_{\mathcal{K}}$, $\lambda_1, \ldots, \lambda_{\mathcal{K}}$. Suppose that there exists a routing policy $P$ such that the process $\tilde{X}(t)$ with the routing policy $P$ is recurrent. Then there exists a non-negative solution of* (2.2) *(and thus for the model with these parameters there exists an ERP).*

We can also rewrite Theorem 2.2 in a different way:

**Corollary 2.1.** *Fix the parameters of the model: $S_1, \ldots, S_{\mathcal{K}}$, $\lambda_1, \ldots, \lambda_{\mathcal{K}}$. Suppose that there is no non-negative solution $\alpha_{ij}$ of the system* (2.2)*. Then for any routing policy $P$, the process $\tilde{X}(t)$ with the routing policy $P$ is transient.*

**Theorem 2.3.** *Fix the parameters of the model: $S_1, \ldots, S_{\mathcal{K}}$, $\lambda_1, \ldots, \lambda_{\mathcal{K}}$. Suppose that there is no positive solution $\alpha_{ij}$ of the system* (2.2)*. Then for any routing policy $P$, the process $\tilde{X}(t)$ with the routing policy $P$ is not positive recurrent.*

The following problem is still open. Fix the parameters of the model: $S_1, \ldots, S_{\mathcal{K}}$, $\lambda_1, \ldots, \lambda_{\mathcal{K}}$. Suppose that there is no positive solution $\alpha_{ij}$ of the system (2.2), but there exists a non-negative solution. Under which conditions on the parameters of the model $S_1, \ldots, S_{\mathcal{K}}$, $\lambda_1, \ldots, \lambda_{\mathcal{K}}$ (and $n$) does there exist a (local) routing policy $P$ such that the process $\tilde{X}(t)$ with the routing policy $P$ is recurrent?

# 3 Proofs

The structure of this section is as follows. First (Section 3.1) we formulate some known fact which we will use in our proofs. In Section 3.2, we introduce some notations and define two functions ($f$ and $g$) we will use to prove Theorem 2.1. Then we prove four lemmas, obtaining bounds on $\mathbf{E}(f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x)$ for JSQ and $\varepsilon$-PSERP. Using these bounds, we prove Theorem 2.1. Then we prove Theorem 2.2. In Section 3.3, we first recall some definitions from complex analysis and apply these to our model. Then we prove Lemma 3.5, which translates the condition of Theorem 2.3 into the language of convex analysis, and then we finish the proof of Theorem 2.3.

## 3.1 Preliminaries

We state some known results that we will use in our proofs. Note that Theorems 3.1 and 3.2 are Theorems 2.2.3 and 2.2.6 respectively from [3], where we use 'positive recurrent' instead of 'ergodic'. This change is necessary as our Markov chains are periodic. That the results also hold for periodic chains is mentioned in Section 1.1 of [3]. In fact, to see that the reformulated theorems are valid it suffices to consider the Markov chain $\eta_\ell$ at embedded instants $\ell = k + pr$, where $p$ is the period of the chain and $k$ is a fixed number.

Let us consider a time homogeneous irreducible Markov chain $\eta_m$ with

countable state space $\mathcal{H}$.

**Theorem 3.1.** *The Markov chain $\eta_m$ is positive recurrent if and only if there exists a positive function $f(x), x \in \mathcal{H}$, a number $\varepsilon > 0$ and a finite set $A \in \mathcal{H}$ such that for every $m$ we have*

$$\begin{aligned} \mathbf{E}(f(\eta_{m+1}) - f(\eta_m) \mid \eta_m = x) &\leq -\varepsilon, \quad x \notin A, \\ \mathbf{E}(f(\eta_{m+1}) \mid \eta_m = x) &< \infty, \quad x \in A. \end{aligned} \tag{3.1}$$

**Theorem 3.2.** *For the Markov chain $\eta_m$ to be not positive recurrent, it is sufficient that there exists a function $f(x), x \in \mathcal{H}$, and constants $C$ and $d$ such that*

- *for every $m$ we have*

  $$\mathbf{E}(f(\eta_{m+1}) - f(\eta_m) \mid \eta_m = x) \geq 0, \ x \in \{f(x) > C\},$$

  *where the sets $\{x \mid f(x) > C\}$ and $\{x \mid f(x) \leq C\}$ are non empty;*

- *for every $m$ we have*

  $$\mathbf{E}(|f(\eta_{m+1}) - f(\eta_m)| \mid \eta_m = x) \leq d, \ x \in \mathcal{H}.$$

The following theorem is an immediate consequence of Theorem 2.1.7 from [3].

**Theorem 3.3.** *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space and $\{\mathcal{F}_n, \ n \geq 0\}$ be an increasing family of $\sigma$-algebras. Let $\{\mathfrak{S}_l, \ l \geq 0\}$ be a sequence of random variables such that $\mathfrak{S}_l$ is $\mathcal{F}_l$-measurable, and $\mathfrak{S}_0$ is a constant. Let*

$$y_{k+1} = \mathfrak{S}_{k+1} - \mathfrak{S}_k.$$

*If there exist positive numbers $\varepsilon$, $M$, such that for each $k$ we have*

$$\mathbf{E}[y_{k+1} \mid \mathcal{F}_k] \leq -\varepsilon, \ \text{a.s.}$$

$$|y_{k+1}| < M \ \text{a.s.},$$

*then, for any $\delta_1 < \varepsilon$, there exist constants $C = C(\mathfrak{S}_0)$ and $\delta_2 > 0$, such that, for any $m > 0$,*

$$\mathbf{P}[\mathfrak{S}_m > -\delta_1 m] < Ce^{-\delta_2 m}.$$

## 3.2 Proofs of Theorems 2.1 and 2.2

To prove Theorem 2.1, we need some additional notations and four lemmas.

Suppose that we are using either JSQ routing policy or $\varepsilon$-PSERP to construct the process $X^e(m)$ (for now, it does not matter which one). We are going to construct a supermartingale with bounded jumps, that will allow us to obtain exponential bounds on $\tau$ (see (2.6) for the definition of $\tau$) and thus to prove positive recurrence of $\tilde{X}^e(m)$.

Let

$$f(X^e(m)) = f(X_1^e(m), \dots, X_n^e(m)) = \sum_{i=1}^{n} (X_i^e(m) - M(m))^2 = \mathfrak{D}(\tilde{X}^e(m)),$$

where $\mathfrak{D}(\tilde{X}^e(m)$ is the shape magnitude defined in (2.7) and

$$g(\tilde{X}^e(m)) = \sqrt{f(X^e(m))} = \left[\sum_{i=1}^{n} (X_i(m) - M(m))^2\right]^{1/2}.$$

We will prove that $g(\tilde{X}^e(m))$ is a supermartingale with bounded jumps. To do that, we will need some auxiliary lemmas. In Lemmas 3.1 and 3.2 we estimate $\mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x]$ in terms of $X^e(m)$ for $\varepsilon$-PSERP and JSQ respectively. In Lemma 3.3 we obtain a bound on $|f(X^e(m+1)) - f(X^e(m))|$, which is needed for the proof that $g(\tilde{X}^e(m))$ has bounded jumps.

First, we introduce the process $(Y_1(m), \dots, Y_n(m))$ obtained when the item that arrives at $S_i$ is directed to node $s_j^i$ with probability $p_j^{(i)} = \alpha_{ij}/\lambda_i$, $j = 1, \dots \kappa_i$ (that is, using SERP). The processes $X^e(m)$ and $Y(m)$ are defined in the same probability space, use the same arrivals, and if $X^e(m) = Y(m) = x$, then $X^e(m+1)$ and $Y(m+1)$ are obtained from $x$ using the respective routing policies.

Using the fact that $\alpha_{ij}$'s are such that arriving items are routed to node $i$ with probability $1/n$ for any $i$, we have

$$\mathbf{E}[(Y_i(m+1) - M(m+1))^2 - (Y_i(m) - M(m))^2 \mid Y(m)]$$

$$
\begin{aligned}
&= \frac{1}{n}\Big(\Big(Y_i(m) + 1 - M(m) - \frac{1}{n}\Big)^2 - (Y_i(m) - M(m))^2\Big) \\
&\quad + \frac{n-1}{n}\Big(\Big(Y_i(m) - M(m) - \frac{1}{n}\Big)^2 - (Y_i(m) - M(m))^2\Big) \\
&= \frac{1}{n} - \frac{1}{n^2},
\end{aligned}
\tag{3.2}
$$

as $M(m+1) = M(m) + \frac{1}{n}$. Thus,

$$
\mathbf{E}[f(Y(m+1)) - f(Y(m)) \mid Y(m)] = n\Big(\frac{1}{n} - \frac{1}{n^2}\Big) = 1 - \frac{1}{n}. \tag{3.3}
$$

Denote by $C_i$ the event that an item arrives at set $S_i$. Recall (2.4) and (2.5). From now on, instead of writing $s^i_{j_{\max}}(X^e(m))$ and $s^i_{j_{\min}}(X^e(m))$, we will write just $s^i_{j_{\max}}$ and $s^i_{j_{\min}}$.

**Lemma 3.1.** *Suppose that the process $X^e(m)$ is constructed using $\varepsilon$-PSERP. Then*

$$
\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x] \\
&\quad = -2\varepsilon \sum_{i=1}^{\mathcal{K}} (X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m)) + 1 - \frac{1}{n}.
\end{aligned}
\tag{3.4}
$$

*Proof of Lemma 3.1.* Suppose $|S_i| > 1$. We have

$$
\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(Y(m+1)) \mid X^e(m) = Y(m) = x,\ C_i] \\
&= \mathbf{E}\Big[\sum_{j=1}^{\kappa_i} \tilde{X}^e_{s^i_j}(m+1)^2 - \tilde{Y}_{s^i_j}(m+1)^2 \mid X^e(m) = Y(m) = x,\ C_i\Big] \\
&= \frac{\varepsilon}{\lambda_i}\Big(\Big(X^e_{s^i_{j_{\min}}}(m) + 1 - M(m) - \frac{1}{n}\Big)^2 \\
&\qquad + \sum_{j \neq j_{\min}} \Big(X^e_{s^i_j}(m) - M(m) - \frac{1}{n}\Big)^2\Big) \\
&\quad -\frac{\varepsilon}{\lambda_i}\Big(\Big(X^e_{s^i_{j_{\max}}}(m) + 1 - M(m) - \frac{1}{n}\Big)^2 \\
&\qquad + \sum_{j \neq j_{\min}} \Big(X^e_{s^i_j}(m) - M(m) - \frac{1}{n}\Big)^2\Big) \\
&= -\frac{2\varepsilon}{\lambda_i}(X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m))
\end{aligned}
\tag{3.5}
$$

as we conditioned on $X^e(m) = Y(m) = x$. Thus,

$$\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x,\ C_i] \\
&\quad= \mathbf{E}[f(X^e(m+1)) - f(Y(m+1)) \mid X^e(m) = Y(m) = x,\ C_i] \\
&\qquad+\mathbf{E}[f(Y(m+1)) - f(X^e(m)) \mid X^e(m) = Y(m) = x,\ C_i] \qquad (3.6)\\
&\quad= -\frac{2\varepsilon}{\lambda_i}(X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m)) \\
&\qquad+\mathbf{E}[f(Y(m+1)) - f(Y(m)) \mid Y(m) = x,\ C_i]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x] \\
&\quad= \sum_{i=1}^{\mathcal{K}} \lambda_i \mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x,\ C_i] \\
&\quad= -2\varepsilon \sum_{i=1}^{\mathcal{K}} (X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m)) + 1 - \frac{1}{n}. \qquad (3.7)
\end{aligned}$$

Note that if there is a neighborhood of size 1, by Condition 1.1 it should be subset of another neighborhood, of size at least 2. As the terms corresponding to neighborhoods of size 1 in (3.7) will be equal to 0, the equation (3.7) still holds. Lemma 3.1 is proved. ■

**Lemma 3.2.** *Suppose that the process $X^e(m)$ is constructed using JSQ routing policy. Then*

$$\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(X^e(m)) \mid X^e(m) = x] \\
&\quad= -2 \sum_{i=1}^{\mathcal{K}} \sum_{j \neq j_{\min}} \alpha_{ij} (X^e_{s^i_j}(m) - X^e_{s^i_{j_{\min}}}(m)) + 1 - \frac{1}{n}. \qquad (3.8)
\end{aligned}$$

*Proof of Lemma 3.2.* Analogously to (3.5),

$$\begin{aligned}
&\mathbf{E}[f(X^e(m+1)) - f(Y(m+1)) \mid X^e(m) = Y(m) = x,\ C_i] \\
&\quad= \sum_{j \neq j_{\min}} \frac{\alpha_{ij}}{\lambda_i} \Big( \Big( X^e_{s^i_{j_{\min}}}(m) + 1 - M(m) - \frac{1}{n} \Big)^2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j''\neq j_{\min}}\Big(X^e_{s^i_{j''}}(m)-M(m)-\frac{1}{n}\Big)^2\\
&-\Big(\Big(Y_{s^i_j}(m)+1-M(m)-\frac{1}{n}\Big)^2+\sum_{j'\neq j}\Big(Y_{s^i_{j'}}(m)-M(m)-\frac{1}{n}\Big)^2\Big)\Big)\\
=\;&\sum_{j\neq j_{\min}}\frac{\alpha_{ij}}{\lambda_i}\Big(\Big(X^e_{s^i_{j_{\min}}}(m)+1-M(m)-\frac{1}{n}\Big)^2+\Big(X^e_{s^i_j}(m)-M(m)-\frac{1}{n}\Big)^2\\
&-\Big(\Big(Y_{s^i_j}(m)+1-M(m)-\frac{1}{n}\Big)^2+\Big(Y_{s^i_{j_{\min}}}(m)-M(m)-\frac{1}{n}\Big)^2\Big)\Big)\\
=\;&-\sum_{j\neq j_{\min}}\frac{2\alpha_{ij}}{\lambda_i}(X^e_{s^i_j}(m)-X^e_{s^i_{j_{\min}}}(m)). \qquad (3.9)
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mathbf{E}[f(X^e(m+1))-f(X^e(m))\mid X^e(m)=x,\ C_i]\\
&\quad=\;-\sum_{j\neq j_{\min}}\frac{2\alpha_{ij}}{\lambda_i}(X^e_{s^i_j}(m)-X^e_{s^i_{j_{\min}}}(m))\\
&\qquad\quad+\mathbf{E}[f(Y(m+1))-f(Y(m))\mid Y(m)=x,\ C_i] \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}[f(X^e(m+1))-f(X^e(m))\mid X^e(m)=x]\\
&\quad=\;\sum_{i=1}^{\mathcal{K}}\lambda_i\mathbf{E}[f(X^e(m+1))-f(X^e(m))\mid X^e(m)=x,\ C_i]\\
&\quad=\;-2\sum_{i=1}^{\mathcal{K}}\sum_{j\neq j_{\min}}\alpha_{ij}(X^e_{s^i_j}(m)-X^e_{s^i_{j_{\min}}}(m))+1-\frac{1}{n}. \qquad (3.11)
\end{aligned}
$$

Lemma 3.2 is proved. ■

Denote by $e_i$ the $i$-th coordinate vector, $i=1,\dots,n$. The next lemma will be used to bound jumps in $f$ due to any possible one-step changes to $x$.

**Lemma 3.3.** *Let $x\in\mathbb{N}^n$ and $m(x)=\frac{1}{n}\sum_{j=1}^n x_j$. If $\sum_{i=1}^n\big(x_i-m(x)\big)^2>0$, then for each $e_i$, $i=1,\dots,n$,*

$$
|f(x+e_i)-f(x)|\le 4\sqrt{f(x)}. \qquad (3.12)
$$

*Proof of Lemma 3.3.* Without loss of generality, consider the first coordinate vector $e_1$. We have then

$$
\begin{aligned}
& f(x+e_1) \\
&= \Big(x_1 + 1 - m(x) - \frac{1}{n}\Big)^2 + \sum_{i=2}^{n} \Big(x_i - m(x) - \frac{1}{n}\Big)^2 \\
&= \big(x_1 - m(x)\big)^2 + 2\big(x_1 - m(x)\big)\Big(1 - \frac{1}{n}\Big) + \Big(1 - \frac{1}{n}\Big)^2 \\
&\quad + \sum_{i=2}^{n} \big(x_i - m(x)\big)^2 - \frac{2}{n}\sum_{i=2}^{n} \big(x_i - m(x)\big) + \frac{n-1}{n^2} \\
&= \sum_{i=1}^{n} \big(x_i - m(x)\big)^2 + 2\big(x_1 - m(x)\big) + 1 - \frac{1}{n} \qquad (3.13)
\end{aligned}
$$

as

$$\frac{1}{n}\sum_{i=1}^{n} \big(x_i - m(x)\big) = 0.$$

Hence for each $i = 1, \ldots, n$ we have

$$f(x+e_i) - f(x) = 2\big(x_i - m(x)\big) + 1 - \frac{1}{n}. \qquad (3.14)$$

It remains to show that, if $\sum_{i=1}^{n} \big(x_i - m(x)\big)^2 > 0$, then

$$\Big|2\big(x_i - m(x)\big) + 1 - \frac{1}{n}\Big| \leq 4\sqrt{f(x)}.$$

Note that $\sum_{i=1}^{n} \big(x_i - m(x)\big)^2 > 0$ implies that there exists $i, j$ such that $|x_i - x_j| \geq 1$. Thus, there exists at least one $l$ such that $|x_l - m(x)| \geq 1/2$ which implies that $\sqrt{f(x)} \geq 1/2$. So,

$$
\begin{aligned}
|f(x+e_i) - f(x)| &\leq 2|x_i - m(x)| + 1 \\
&\leq 2\Big[\sum_{i=1}^{n} \big(x_i - m(x)\big)^2\Big]^{1/2} + 1 \\
&\leq 4\sqrt{f(x)}.
\end{aligned}
$$

Lemma 3.3 is proved. ∎

Lemma 3.3 implies that, for any RP, if $\sum_{i=1}^{n}\big(X_i^e(m)-M(m)\big)^2>0$, then

$$|f(X^e(m+1))-f(X^e(m))|\le 4\sqrt{f(X^e(m))}=4g(\tilde{X}^e(m)). \tag{3.15}$$

It is important to note that the next computations are valid for JSQ and for $\varepsilon$-PSERP.

**Lemma 3.4.** *There exist $c_2>0$ and $a>0$, such that for all $x\in\mathbb{N}^n$ with*

$$\max_{i=1,\dots,n}\Big|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\Big|\ge a$$

*it holds that*

$$\mathbf{E}(f(X^e(m+1))-f(X^e(m))\mid X^e(m)=x)\le -c_2\sqrt{f(x)}. \tag{3.16}$$

*Proof of Lemma 3.4.* We have

$$\begin{aligned}
f(X^e(m)) &= \sum_{l=1}^{n}\big(X_l^e(m)-M(m)\big)^2\\
&\le \sum_{i=1}^{\mathcal{K}}\sum_{j\in S_i}(X_j^e(m)-M(m))^2\\
&\le \sum_{i=1}^{\mathcal{K}}|S_i|\ \max_{j\in S_i}\{(X_j^e(m)-M(m))^2\}\\
&\le n\sum_{i=1}^{\mathcal{K}}\max_{j\in S_i}(X_j^e(m)-M(m))^2.
\end{aligned}$$

We now show that under Condition 1.1 we have

$$\sum_{i=1}^{\mathcal{K}}\max_{j\in S_i}(X_j^e(m)-M(m))^2\le c_3\Big(\sum_{i=1}^{\mathcal{K}}(X^e_{s^i_{j_{\max}}}(m)-X^e_{s^i_{j_{\min}}}(m))\Big)^2 \tag{3.17}$$

and also

$$\sum_{i=1}^{\mathcal{K}}\max_{j\in S_i}(X_j^e(m)-M(m))^2\le c_4\Big(\sum_{i=1}^{\mathcal{K}}\sum_{j\ne j_{\min}}\alpha_{ij}(X^e_{s^i_j}(m)-X^e_{s^i_{j_{\min}}}(m))\Big)^2. \tag{3.18}$$

Let us consider (3.17). If $X^e_{s^i_{j_{\min}}}(m) \le M(m) \le X^e_{s^i_{j_{\max}}}(m)$, then, obviously, $(X^e_j(m) - M(m))^2 \le (X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m))^2$. Suppose that $M(m) < X^e_{s^i_{j_{\min}}}(m)$ (the case $M(m) > X^e_{s^i_{j_{\max}}}(m)$ can be treated analogously). Consider the sets of nodes $\{j : X^e_j(m) \le M(m)\}$ and $\{j : X^e_j(m) > M(m)\}$. By Condition 1.1 some neighbourhood contains nodes from each of these sets and hence there exists $i^*$ such that $X^e_{s^{i^*}_{j_{\min}}}(m) \le M(m) \le X^e_{s^{i^*}_{j_{\max}}}(m)$ and a sequence of neighbourhoods indexed by $i_0 = i, i_1, \ldots, i_k = i^*$ such that $S_{i_{l-1}} \cap S_{i_l} \ne \emptyset$, $l = 1, \ldots, k$. Thus,

$$\begin{aligned} \max_{j \in S_i} |(X^e_j(m) - M(m)| &\le X^e_{s^i_{j_{\max}}}(m) - M(m) \\ &\le X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m) + X^e_{s^i_{j_{\min}}}(m) - M(m) \\ &\le X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m) + X^e_{s^{i_1}_{j_{\max}}}(m) - M(m). \end{aligned} \tag{3.19}$$

The last inequality is due to the fact that, as $S_i \cap S_{i_1} \ne \emptyset$, it holds

$$\begin{aligned} X^e_{s^i_{j_{\min}}}(m) &= \min_{s^i_j \in S_i} X^e_{s^i_j}(m) \le \min_{s^i_j \in S_i \cap S_{i_1}} X^e_{s^i_j}(m) \\ &\le \max_{s^i_j \in S_i \cap S_{i_1}} X^e_{s^i_j}(m) \le \max_{s^{i_1}_j \in S_{i_1}} X^e_{s^{i_1}_j}(m) = X^e_{s^{i_1}_{j_{\max}}}(m) \end{aligned}$$

Continuing (3.19), we get

$$\begin{aligned} \max_{j \in S_i} |(X^e_j(m) - M(m)| &\le X^e_{s^i_{j_{\max}}}(m) - M(m) \\ &\le X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m) + X^e_{s^{i_1}_{j_{\max}}}(m) - M(m) \\ &\le X^e_{s^i_{j_{\max}}}(m) - X^e_{s^i_{j_{\min}}}(m) + X^e_{s^{i_1}_{j_{\max}}}(m) - X^e_{s^{i_1}_{j_{\min}}}(m) \\ &\quad + X^e_{s^{i_2}_{j_{\max}}}(m) - M(m) \end{aligned}$$

and so on until $i_k = i^*$ (at the last step one has to use $X^e_{s^{i^*}_{j_{\min}}}(m) \le M(m)$). So, we obtain

$$\max_{j \in S_i} |(X^e_j(m) - M(m)| \le \sum_{l=0}^{k} (X^e_{s^{i_l}_{j_{\max}}}(m) - X^e_{s^{i_l}_{j_{\min}}}(m))$$

and (3.17) follows with some $c_3 \le \kappa$. The argument for (3.18) is similar.

Then Lemma 3.1 together with (3.17) (for $\varepsilon$-PSERP), and Lemma 3.2 together with (3.18) (for JSQ), imply that, for some $c_2 > 0$,

$$\mathbf{E}(f(X^e(m+1)) - f(X^e(m) \mid X^e(m)) \le -c_2\sqrt{f(X^e(m))} = -c_2 g(\tilde{X}^e(m)), \tag{3.20}$$

when $f(X^e(m))$ is large enough. Lemma 3.4 is proved. ∎

*Proof of Theorem 2.1.* First, we verify that $g(\tilde{X}^e(m))$ has bounded jumps. If $\sum_{i=1}^n \left(X_i^e(m) - M(m)\right)^2 = 0$, then, obviously, $g(\tilde{X}^e(m+1)) - g(\tilde{X}^e(m)) \le \text{const}$. So, suppose that $\sum_{i=1}^n \left(X_i^e(m) - M(m)\right)^2 > 0$.

Using inequality $|\sqrt{1+b} - 1| \le |b|$ for $b \ge -1$, we obtain that

$$\begin{aligned}
&|g(\tilde{X}^e(m+1)) - g(\tilde{X}^e(m))| \\
&= [f(X^e(m))]^{1/2}\left|\left[1 + \frac{f(X^e(m+1)) - f(X^e(m))}{f(X^e(m))}\right]^{1/2} - 1\right| \\
&\le [f(X^e(m))]^{1/2}\left|\frac{f(X^e(m+1)) - f(X^e(m))}{f(X^e(m))}\right| \\
&= \frac{|f(X^e(m+1)) - f(X^e(m))|}{[f(X^e(m))]^{1/2}} \\
&\le 4,
\end{aligned} \tag{3.21}$$

by Lemma 3.3.

Let

$$A = F(\mathbb{N}^n) \cap \{x \in \mathbb{R}^n : \max_i |x_i| < a\},$$

where $a$ is from Lemma 3.4. That is, $A$ is the set of possible configurations of $\tilde{X}^e$ such that $\max_{i=1,\dots,n} |\tilde{X}_i^e| < a$. Note that the set $A$ is finite. Let us now prove that

$$\mathbf{E}[g(\tilde{X}^e(m+1)) - g(\tilde{X}^e(m)) \mid \tilde{X}^e(m) = x] \le -c_2/\sqrt{2},$$

if $x \notin A$. Indeed, if $x \in F(\mathbb{N}^n) \setminus A$, as $\sqrt{1+b} \le 1 + \frac{b}{2}$ for $b \ge -1$, we get (using Lemma 3.4)

$$\mathbf{E}[g(\tilde{X}^e(m+1)) - g(\tilde{X}^e(m)) \mid \tilde{X}^e(m) = x]$$

$$
\begin{aligned}
&= \mathbf{E}\left[[f(\tilde{X}^e(m))]^{1/2}\Big(\left[1+\frac{f(\tilde{X}^e(m+1))-f(\tilde{X}^e(m))}{f(\tilde{X}^e(m))}\right]^{1/2}-1\Big)\mid \tilde{X}^e(m)=x\right] \\
&\leq \frac{\mathbf{E}[f(\tilde{X}^e(m+1))-f(x)\mid \tilde{X}^e(m)=x]}{2\sqrt{f(x)}} \qquad (3.22)\\
&\leq -\frac{c_2}{2}.
\end{aligned}
$$

Thus, by Theorem 3.1 the process $\tilde{X}^e$ is positive recurrent.

For $\tau_A = \inf\{m > 0 : \tilde{X}^e(m+k) \in A\}$, take now

$$
\mathfrak{S}_m = \begin{cases} g(\tilde{X}^e(m)), & \text{if} \quad m \leq \tau_A, \\ -(m-\tau_A), & \text{if} \quad m > \tau_A \end{cases}
$$

and apply Theorem 3.3 to the sequence $\{\mathfrak{S}_m\}$. We have that for any $\delta_1 < c_2/2$, there exist $C$ and $\delta_2$ such that

$$
\mathbf{P}[\tau_A > (1-\delta_1)m \mid \tilde{X}^e(k) = x' \notin A] < Ce^{-\delta_2 m}.
$$

Note that there exist $k$ and $\delta > 0$ such that for any $y \in A$

$$
\mathbf{P}[\tilde{X}^e(m+l) = 0 \text{ for some } l \leq k \mid \tilde{X}^e(m) = y] \geq \delta.
$$

It is then not difficult to obtain that $\mathbf{E}(e^{c'\tau} \mid \tilde{X}^e(k) = x) < \infty$, where $\tau = \inf\{m > 0 : \tilde{X}^e(m) = 0\}$. Theorem 2.1 is proved. ■

*Proof of Theorem 2.2.* Let $N_i(t)$ be the number of arrivals at $S_i$ by time $t$. Since $N_i(t)$ is a Poisson process with rate $\lambda_i$, a.s. $N_i(t) \to \infty$ and $N_i(t)/t \to \lambda_i$ as $t \to \infty$, $i = 1, \ldots, n$.

As $\tilde{X}(t)$ is recurrent, we have that for almost every realization of the process $\tilde{X}(t)$ there exists an infinite sequence $t_1, t_2, \ldots$ such that $\tilde{X}(t_j) = 0$ for all $j$. For these moments $t_j$ we can define

$$
\alpha_{ik}(t_j) = \frac{\lambda_i N_{ik}(t_j)}{N_i(t_j)},
$$

where $N_{ik}(t_j)$ is the number of items arrived at $S_i$ and routed to node $s^i_k$ by time $t_j$. So, sending the proportion $\frac{\alpha_{ik}(t_j)}{\lambda_i}$ of items arriving at $S_i$ to

$s^i_k$, results in the same number of items at all nodes. As the sequence of $\alpha_{ik}(t_j)$ is bounded, we can chose a subsequence $\alpha_{ik}(t_{j'}) \to \alpha_{ik}$, as $t_{j'} \to \infty$. Evidently, $\alpha_{ik} \geq 0$ and $\sum_{k=1}^{\kappa_i} \alpha_{ik} = \lambda_i$. Then, as

$$X_l(t_j) = \frac{1}{n} \sum_{l'=1}^{n} X_{l'}(t_j) = \frac{1}{n} \sum_{i=1}^{\mathcal{K}} N_i(t_j)$$

we obtain

$$\begin{aligned}
\frac{1}{n} &= \frac{X_l(t_j)}{\sum_{i=1}^{\mathcal{K}} N_i(t_j)} \\
&= \frac{1}{\sum_{i=1}^{\mathcal{K}} N_i(t_j)} \sum_{i=1}^{\mathcal{K}} \sum_{m=1}^{\kappa_i} N_{im}(t_j) \delta_{l,s^i_m} \\
&= \frac{1}{\sum_{i=1}^{\mathcal{K}} N_i(t_j)} \sum_{i=1}^{\mathcal{K}} N_i(t_j) \sum_{m=1}^{\kappa_i} \frac{N_{im}(t_j)}{N_i(t_j)} \delta_{l,s^i_m} \\
&= \frac{t_j}{\sum_{i=1}^{\mathcal{K}} N_i(t_j)} \sum_{i=1}^{\mathcal{K}} \frac{N_i(t_j)}{t_j} \sum_{m=1}^{\kappa_i} \frac{\alpha_{im}(t_j)}{\lambda_i} \delta_{l,s^i_m}.
\end{aligned}$$

As

$$\frac{N_i(t)}{t} \to \lambda_i \text{ and } \frac{t}{\sum_{i=1}^{\mathcal{K}} N_i(t)} \to \frac{1}{\sum_{i=1}^{\mathcal{K}} \lambda_i} = 1,$$

and $\alpha_{im}(t_{j'}) \to \alpha_{im}$, we see that $\{\alpha_{im}\}$ is indeed a solution of (2.2) and Theorem 2.2 is proved. ■

### 3.3 Proof of Theorem 2.3

We will use theorems from [7], therefore let us recall some definitions from there. A subset $C$ in $\mathbb{R}^n$ is called *convex* if $(1-\lambda)x + \lambda y \in C$ for every $x \in C$, $y \in C$ and $0 < \lambda < 1$. A subset $M$ in $\mathbb{R}^n$ is called an *affine set* if $(1-\lambda)x + \lambda y \in M$ for every $x \in M$, $y \in M$ and $\lambda \in \mathbb{R}$. Given any set $A \subset \mathbb{R}^n$ there exists a unique smallest affine set containing $A$ (namely, the intersection of the collection of the affine sets $M$ such that $A \subset M$), this set is called *affine hull* of $A$ and is denoted by $\operatorname{aff} A$. Given a set $A \subset \mathbb{R}^n$ the interior that results when $A$ is regarded as a subset of its affine hull $\operatorname{aff} A$

is called *relative interior* of $A$ and is denoted by $\operatorname{ri} A$. The closure of $A$ is denoted by $\operatorname{cl} A$. Note that $\operatorname{cl}(\operatorname{cl} A) = \operatorname{cl} A$ and $\operatorname{ri}(\operatorname{ri} A) = \operatorname{ri} A$; moreover, if $A$ is convex, then $\operatorname{cl}(\operatorname{ri} A) = \operatorname{cl} A$ (see Theorem 6.3 in [7]). If $A$ is convex and $A \neq \varnothing$, then $\operatorname{ri} A \neq \varnothing$ (see Theorem 6.2 in [7]). A set $A$ is called *relatively open* if $\operatorname{ri} A = A$.

Let us apply the definitions to our model. Note that $\Lambda_i \in \mathbb{R}^{\kappa}_i$ is convex, $i = 1, \ldots, \mathcal{K}$. By $E$ denote the linear transformation that takes a point $p = (p^{(1)}, \ldots, p^{(\mathcal{K})}) \in \mathbb{R}^{\kappa_1 + \cdots + \kappa_{\mathcal{K}}}$ to the point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$x_\ell = \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \delta_{\ell, s_j^i} \quad \text{for } \ell = 1, \ldots, n$$

where, as before,

$$\delta_{\ell m} = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m. \end{cases}$$

$$\begin{aligned} \text{Let } L &:= E(\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}) \subset \mathbb{R}^n \text{ and} \\ D &:= F(L) \subset \mathfrak{M} \subset \mathbb{R}^n. \end{aligned} \tag{3.23}$$

Since, for $i = 1, \ldots, \mathcal{K}$, the set $\Lambda_i$ is convex, we see that the set $\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}$ is convex (see Theorem 3.5 in [7]). As $E$ and $F$ are linear transformations, the sets $L$ and $D$ are convex (see Theorem 3.4 in [7]). Since, for $i = 1, \ldots, \mathcal{K}$, the set $\Lambda_i$ is compact, the set $\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}$ is also compact. Since $E$ and $F$ are linear transformations, and therefore, continuous transformations, we see that the sets $L$ and $D$ are compact. In particular, $D$ is closed, that is, $\operatorname{cl} D = D$.

To translate the condition of Theorem 2.3 to the language of convex analysis, we need the following lemma.

**Lemma 3.5.** *For any parameters of the model $S_1, \ldots, S_{\mathcal{K}}$ and $\lambda_1, \ldots, \lambda_n$, the following statements are equivalent:*

1. $0 \in \operatorname{ri} D$;

*2. there exists a positive solution $\alpha_{ij}$ of the system* (2.2).

*Proof.* Note that $\operatorname{ri}\Lambda_i$ is the set of points $p^{(i)} = (p_1^{(i)}, \ldots, p_{\kappa_i}^{(i)}) \in \mathbb{R}^{\kappa_i}$ such that

$$\begin{cases} p_j^{(i)} > 0 & \text{for } j = 1, \ldots, \kappa_i, \\ \sum_{j=1}^{\kappa_i} p_j^{(i)} = 1 \end{cases} \tag{3.24}$$

Moreover,

$$\operatorname{ri}(\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}) = (\operatorname{ri}\Lambda_1) \times \cdots \times (\operatorname{ri}\Lambda_{\mathcal{K}}) \tag{3.25}$$

(see the proof of Corollary 6.6.1 in [7]). Since $F \circ E$ is a linear transformation, we see that

$$F(E(\operatorname{ri}(\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}))) = \operatorname{ri} F(E(\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}})) = \operatorname{ri} D$$

(for the first equality see Theorem 6.6. in [7]).

Thus we have proved that $F(E((\operatorname{ri}\Lambda_1) \times \cdots \times (\operatorname{ri}\Lambda_{\mathcal{K}}))) = \operatorname{ri} D$. Therefore, $y = F(E(p)) \in \operatorname{ri} D$ if and only if $p \in (\operatorname{ri}\Lambda_1) \times \cdots \times (\operatorname{ri}\Lambda_{\mathcal{K}})$. Recalling (3.24) for $\operatorname{ri}\Lambda_i$, we get that $y = F(E(p)) \in \operatorname{ri} D$ if and only if

$$\begin{cases} p_j^{(i)} > 0 & \text{for } j = 1, \ldots, \kappa_i \text{ and } i = 1, \ldots, \mathcal{K}, \\ \sum_{j=1}^{\kappa_i} p_j^{(i)} = 1 & \text{for } i = 1, \ldots, \mathcal{K}. \end{cases} \tag{3.26}$$

Suppose that item 1 holds, that is, $0 \in \operatorname{ri} D$. Then there exists $p \in \mathbb{R}^{\kappa_1 + \cdots + \kappa_{\mathcal{K}}}$ and $x \in \mathbb{R}^n$ such that $p$ satisfies (3.26), $E(p) = x$ and $F(x) = 0$. Then we have

$$\sum_{\ell=1}^{n} x_\ell = \sum_{\ell=1}^{n} \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \delta_{l, s_j^i} = \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} = \sum_{i=1}^{\mathcal{K}} \lambda_i = 1.$$

In the first equation we use the definition of $E$, in the second the fact that $\sum_{\ell=1}^{n} \delta_{l, s_j^i} = 1$, in the third the second line from (3.26). Therefore, using the definition of $F$, it follows from $F(x) = 0$ that $x = E(p) = (\frac{1}{n}, \ldots, \frac{1}{n})$. Then $\sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \delta_{l, s_j^i} = \frac{1}{n}$ for $\ell = 1, \ldots, n$. We have proved that if $0 \in \operatorname{ri} D$ then

there exists $p \in \mathbb{R}^{\kappa_1+\cdots+\kappa_{\mathcal{K}}}$ such that

$$\begin{cases} p_j^{(i)} > 0 & \text{for } j = 1, \ldots, \kappa_i \text{ and } i = 1, \ldots, \mathcal{K}, \\ \sum_{j=1}^{\kappa_i} p_j^{(i)} = 1 & \text{for } i = 1, \ldots, \mathcal{K}, \\ \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \delta_{l,s_j^i} = \frac{1}{n} & \text{for } \ell = 1, \ldots, n. \end{cases} \tag{3.27}$$

Substituting $\frac{\alpha_{ij}}{\lambda_i}$ for $p_j^{(i)}$ in (3.27), we get a positive solution of (2.2). Thus item 2 holds.

Now suppose that item 2 holds, that is, there exists $p \in \mathbb{R}^{\kappa_1+\cdots+\kappa_{\mathcal{K}}}$ that satisfies (3.27). Let us prove that $0 \in \operatorname{ri} D$. Comparing the first and second line of (3.27) with (3.26), we get $F(E(p)) \in \operatorname{ri} D$. Let $x = E(p)$. Then

$$x_\ell = \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \delta_{l,s_j^i} = \frac{1}{n}.$$

In the first equation we use the definition of $E$ and in the second the third line of (3.27). From the definition of $F$ it follows that $F(x) = 0$. Therefore, $F(E(p)) = 0$. Thus $0 = F(E(p)) \in \operatorname{ri} D$ and item 1 holds. ■

Let us recall some additional definitions from [7]. For $M \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, the *translate* of $M$ by $a$ is defined to be set

$$M + a = \{x + a \mid x \in M\}.$$

A translate of an affine set is another affine set. An affine set $M$ is *parallel* to an affine set $L$ if $M = L + a$ for some $a$. Each non-empty affine set is parallel to a unique subspace $L$ (see Theorem 1.2 in [7]). The *dimension* of a non-empty affine set is defined as the dimension of the subspace parallel to it. An $(n-1)$-dimensional affine set in $\mathbb{R}^n$ is called a *hyperplane*. By $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^n$: $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Given $\beta \in \mathbb{R}$ and a non-zero $b \in \mathbb{R}^n$, the set

$$H = \{x \mid \langle x, b \rangle = \beta\}$$

is a hyperplane in $\mathbb{R}^n$; moreover, every hyperplane may be represented in this way, with $b$ and $\beta$ unique up to common non-zero multiple (see Theorem 1.3 in [7]).

For any non-zero $b \in \mathbb{R}^n$ and any $\beta \in \mathbb{R}$, the sets

$$\{x \mid \langle x, b\rangle \leq \beta\}, \qquad \{x \mid \langle x, b\rangle \geq \beta\}$$

are called *closed half-spaces*. The sets

$$\{x \mid \langle x, b\rangle < \beta\}, \qquad \{x \mid \langle x, b\rangle > \beta\}$$

are called *open half-spaces*. The half-spaces depend only on the hyperplane $H = \{x : \langle x, b\rangle = \beta\}$. One may speak unambiguously, therefore, of the open and closed hyperspaces corresponding to a given hyperplane.

Let $C_1$ and $C_2$ be non-empty sets in $\mathbb{R}^n$. A hyperplane is said to *separate* $C_1$ and $C_2$ if $C_1$ is contained in one of the closed half spaces associated with $H$ and $C_2$ lies in the opposite half-space. It is said that to separate $C_1$ and $C_2$ properly if $C_1$ and $C_2$ are not *both* actually contained in $H$ itself.

Now we are ready to prove Theorem 2.3. By Lemma 3.5, we have to show that $0 \notin \operatorname{ri} D$.

Note that the one point set $\{0\}$ is an affine set, $\operatorname{ri} D$ is a relatively open convex set and $(\operatorname{ri} D) \cap \{0\} = \varnothing$. Therefore, there exists a hyperplane $H$ containing 0 such that one of the open half-spaces associated with $H$ contains $\operatorname{ri} D$ (see Theorem 11.2 in [7]). Since $0 \in H$, we see that $H = \{x : \langle x, b\rangle = 0\}$ with some $b \in \mathbb{R}^n$, $b \neq 0$. Substituting, if it is necessary, $-b$ for $b$, we see that there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ that sends point $y \in \mathbb{R}^n$ to value $\langle y, b\rangle$, and if $y \in \operatorname{ri} D$, then $f(y) > 0$.

Recall that the state space of the Markov chain $\tilde{X}^e(m)$ is $F(\mathbb{N}^n)$. Since $\operatorname{ri} D \subset \mathfrak{M}$, we see that there is a point $z \in F(\mathbb{N}^n) \subset \mathfrak{M}$ such that $f(z) > 0$. Also, $0 \in F(\mathbb{N}^n)$ and $f(0) = 0$. To prove that $\tilde{X}^e(m)$ is not positive recurrent let us apply Theorem 3.2 to the Markov chain $\tilde{X}^e(m)$ and the function $f$. To apply the theorem, we see that it is enough to check that

$$\mathbf{E}(f(\tilde{X}^e(m+1)) - f(\tilde{X}^e(m)) \mid \tilde{X}^e(m) = z) \geq 0 \quad \text{for any } z \in F(\mathbb{N}^n) \tag{3.28}$$

To prove (3.28) it is enough to prove

$$\mathbf{E}(f(F(X^e(m+1))) - f(F(X^e(m))) \mid X^e(m) = x) \geq 0 \quad \forall x \in \mathbb{N}^n \tag{3.29}$$

To prove (3.29), we need some notation. For $i = 1, \dots, \mathcal{K}$, denote by $e_j^{(i)}$ the $j$-th coordinate vector in $\mathbb{R}^{\kappa_i}$. By $T$ denote the linear transformation that takes a point $p = (p^{(1)}, \dots, p^{(\mathcal{K})}) \in \mathbb{R}^{\kappa_1 + \cdots + \kappa_{\mathcal{K}}}$ to the point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that

$$x_\ell = \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} p_j^{(i)} \delta_{\ell, s_j^i} \quad \text{for } \ell = 1, \dots, n$$

In particular, $T$ takes the point $e_j^{(i)}$ to the point $x$ such that $x_\ell = \delta_{\ell, s_j^i}$, for $l = 1, \dots, n$.

Let us prove (3.29). Take any $x \in \mathbb{N}^n$. Recall that, for routing policy $P$, we have $p = P(x) \in \Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}}$. Moreover,

$$\begin{aligned}
&\mathbf{E}\big(f(F(X^e(m+1))) - f(F(X^e(m))) \mid X^e(m) = x\big) \\
&\quad = \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} \big(f(F(x + T(e_j^{(i)}))) - f(F(x))\big) \\
&\quad = \sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} f\big(F(T(e_j^{(i)}))\big) \\
&\quad = f\big(F(\sum_{i=1}^{\mathcal{K}} \sum_{j=1}^{\kappa_i} \lambda_i p_j^{(i)} T(e_j^{(i)}))\big) = f\big(F(E(p))\big) \geq 0.
\end{aligned}$$

In the second and third equalities we use that $f \circ F$ is a linear functional. Let us check the last inequality. We have

$$F(E(p)) \in F(E(\Lambda_1 \times \cdots \times \Lambda_{\mathcal{K}})) = D.$$

Since $D$ is closed and convex, we see that $\mathrm{cl}(\mathrm{ri}\, D) = \mathrm{cl}\, D = D$ (see the properties of operations ri and cl in the beginning of Section 3.3). Note that $f(y) > 0$ for $y \in \mathrm{ri}\, D$ and linear functional $f$ is continuous, therefore, $f(y) \geq 0$ for $y \in \mathrm{cl}(\mathrm{ri}\, D) = D$.

Thus all conditions of Theorem 3.2 are satisfied and Theorem 2.3 is proved. ■

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